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Entropy Numbers of Diagonal Operators between Symmetric Banach Spaces

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We give the exact order of the dyadic entropy numbers of the identities from l_n^p to l_n^r where $p < r$. Weaker estimates can be found in [3, 4]. The crucial lemma is a combinatorial result from [5]. Then we consider (dyadic) entropy numbers of identities between finite-dimensional symmetric Banach spaces. We obtain a simple expression that gives the exact order up to some logarithmic factor. This allows us to generalize a theorem due to B. Carl [1] about diagonal operators. It turns out that the result still holds under much weaker assumptions on the spaces. More precisely, the assumptions are not so much concerned with the spaces themselves but (what seems to be intuitively clear) with the relation between the spaces.

1. PRELIMINARIES

A basis $\{e_i\}_{i=1}^{\infty}$ of a Banach space E is called symmetric if we have for all permutations π, all sings ε_i, $i \in \mathbb{N}$, and all $a_i \in \mathbb{R}$,

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} \varepsilon_i a_{\pi(i)} e_i \right\|.$$

The biorthogonal functionals are denoted by $\{e_i^*\}_{i=1}^{\infty}$. We put

$$\lambda_E(k) = \left\| \sum_{i=1}^{k} e_i \right\|.$$

A fact that is used frequently is

$$\lambda_E(k)\, \lambda_{E^*}(k) = \left\| \sum_{i=1}^{k} e_i \right\| \left\| \sum_{i=1}^{k} e_i^* \right\| = k.$$

B_E is the unit ball of the space E. We consider the natural identity between

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(finite-dimensional) symmetric (spaces with a symmetric basis) spaces given by

$$\mathrm{id}_{E,F}\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i f_i,$$

where $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are symmetric bases of E and F, respectively. If $E = l_n^p$ and $F = l_n^q$ we might also write $\mathrm{id}_{p,q}$.

The kth entropy number of an operator $S \in L(E, F)$ is

$$\varepsilon_k(S) = \inf\left\{\sigma \mid S(B_E) \subseteq \bigcup_{i=1}^{k} \{x_i + \sigma B_F\},\ x_i \in F\right\}$$

and the kth dyadic entropy number is

$$\mathrm{ent}_k(S) = \inf\left\{\sigma \mid S(B_E) \subseteq \bigcup_{i=1}^{2^k} \{x_i + \sigma B_F\},\ x_i \in F\right\}.$$

The norm of the Lorentz space $l^{s,t}$, $0 < s$, $t \leqslant \infty$, is given by

$$\|x\|_{s,t} = \left(\sum_{k=1}^{\infty} |k^{1/s-1/t}\xi_k|^t\right)^{1/t}.$$

Also, we are concerned with the operator ideal $\mathscr{L}_{s,t}^{(e)}(E, F)$ equipped with a quasi-norm that is equivalent to [4]

$$\|(\mathrm{ent}_k(S))_{k=1}^{\infty}\|_{s,t}.$$

The measure on $\mathbb{R}^n$ that we use throughout this paper is the usual Lebesgue measure. So, we have for the unit balls B_∞ of l_n^∞ and B_1 of l_n^1 that $\mathrm{vol}(B_\infty) = 2^n$ and $\mathrm{vol}(B_1) = 2^n(1/n!)$.

Entropy Numbers for Symmetric Banach Spaces

THEOREM 1. *Let* $\mathrm{id}_{p,r} \in L(l_n^p, l_n^r)$, $1 \leqslant p < r \leqslant \infty$, *be the natural identity. Then we have*

$$\mathrm{ent}_k(\mathrm{id}_{p,r}) \sim \begin{cases} 1, & \text{if } k \leqslant \log n, \\ \left(\dfrac{\log(n/k+1)}{k}\right)^{1/p-1/r}, & \text{if } \log n \leqslant k \leqslant n, \\ 2^{-k/n} n^{1/r-1/p}, & \text{if } k \geqslant n. \end{cases}$$

We require several lemmas. The first can be found in [5, Ex. 29].

LEMMA 2. *Let $k, n \in \mathbb{N}$.*

$$\operatorname{card}\{x \in Z^n \mid \|x\|_1 \leqslant k\} = \sum_{i=0}^{n} 2^{n-i} \binom{n}{i} \binom{k}{n-i}.$$

The following lemma is contained in [6], and also, implicitly, in [2].

LEMMA 3. *Let $\{e_i\}_{i=1}^n$ be a symmetric, normalized basis of E. Then we have for the volume of the unit ball B_E*

$$\frac{1}{2e} \operatorname{vol}(B_E)^{1/n} \leqslant \left\| \sum_1^n e_i \right\|^{-1} \leqslant \frac{1}{2} \operatorname{vol}(B_E)^{1/n}.$$

Proof. We have

$$\left\| \sum_1^n e_i \right\|^{-1} B_\infty \subseteq B_E \subseteq \left\| \sum_1^n e_i^* \right\| B_1.$$

By taking volumes and by using $\lambda_E(n)\, \lambda_{E^*}(n) = n$ we get

$$\left\| \sum_1^n e_i \right\|^{-1} 2 \leqslant \operatorname{vol}(B_E)^{1/n} \leqslant \left\| \sum_1^n e_i^* \right\| \left(\frac{2^n}{n!} \right)^{1/n} \leqslant 2e \left\| \sum_1^n e_i \right\|^{-1}. \quad \blacksquare$$

LEMMA 4. *Let $\{e_i\}_1^n$ and $\{f_i\}_1^n$ be symmetric, normalized bases of E and F, $\mathrm{id}_{E,F}$ the natural identity. Then we have*

$$e^{-1} 2^{-k/n} \frac{\lambda_F(n)}{\lambda_E(n)} \leqslant \operatorname{ent}_k(\mathrm{id}_{E,F}) \leqslant c 2^{-k/n} \frac{\lambda_F(n)}{\lambda_E(n)} \qquad \text{for} \quad k \geqslant n,$$

where c denotes an absolute constant.

Proof. The left-hand inequality follows by considering the volumes of B_E and B_F. Suppose we have

$$B_E \subseteq \bigcup_{i=1}^{r} \{x_i + \sigma B_F\}.$$

It follows that $\operatorname{vol}(B_E)^{1/n} \leqslant r^{1/n} \sigma \operatorname{vol}(B_F)^{1/n}$. By Lemma 3 we get $\lambda_E(n)^{-1} \leqslant e r^{1/n} \sigma \lambda_F(n)^{-1}$. The left-hand inequality follows.

The right-hand inequality for $E = l_n^1$ and $F = l_n^\infty$ follows from Lemma 2. Since the result is already contained in [3] we don't want to go into further details. The general case follows now from a factorization. Indeed, consider the factorization $\mathrm{id}_{E,1} \in L(E, l_n^1)$, $\mathrm{id}_{1,\infty} \in L(l_n^1, l_n^\infty)$ and $\mathrm{id}_{\infty,F} \in L(l_n^\infty, F)$.

$$\begin{aligned} \operatorname{ent}_k(\mathrm{id}_{E,F}) &\leqslant \operatorname{ent}_k(\mathrm{id}_{1,\infty}) \|\mathrm{id}_{E,1}\| \|\mathrm{id}_{\infty,F}\| \\ &= \lambda_{E^*}(n)\, \lambda_F(n) \operatorname{ent}_k(\mathrm{id}_{1,\infty}). \quad \blacksquare \end{aligned}$$

Proof of Theorem 1. Obviously, the case $k \geqslant n$ is covered by Lemma 4. We turn to the case $k \leqslant \log n$. We have trivially $\operatorname{ent}_k(\mathrm{id}_{p,r}) \leqslant 1$ since $p < r$. On the other hand, since the unit vectors $e_1, \ldots, e_n$ satisfy

$$\|e_i - e_j\|_r = 2^{1/r} \qquad \text{for} \quad i \neq j,$$

we get

$$\operatorname{ent}_k(\mathrm{id}_{p,r}) \geqslant \tfrac{1}{4} \qquad \text{for} \quad k = 1, \ldots, [\log n].$$

Now, we settle the case $[\log n] \leqslant k \leqslant n$. We do this first for $p = 1$ and $r = \infty$. By Lemma 2 we get, for $k \leqslant n$,

$$\varepsilon_{2^k\binom{n}{k}}(\mathrm{id}_{1,\infty}) \geqslant \frac{1}{4k} \qquad \text{and} \qquad \varepsilon_{2^k\binom{n+k}{k}}(\mathrm{id}_{1,\infty}) \leqslant \frac{1}{k},$$

since

$$2^k \binom{n}{k} = 2^k \binom{n}{n-k} \leqslant \sum_{i=0}^{n} 2^{n-i} \binom{n}{i}\binom{k}{n-i}$$
$$\leqslant \sum_{i=0}^{n-k} 2^k \binom{n}{i}\binom{k}{n-i} = 2^k \binom{n+k}{k}.$$

It is left to pass to dyadic entropy numbers: We have for $k \leqslant n$ that $\log(2^k\binom{n}{k}) \sim k \log(n/k + 1)$. Then we have to apply that the inverse function of $k \log(n/k + 1)$ is proportional to $k \log^{-1}(n/k + 1)$. The cases $p \neq 1$ and $r \neq \infty$ follow by interpolation. Indeed, by [4, p. 173] we have

$$\operatorname{ent}_k(\mathrm{id}_{p,r}) \leqslant 4 \operatorname{ent}_k(\mathrm{id}_{1,\infty})^{1/p - 1/r},$$

and by [4, p. 169]

$$\frac{1}{3k} \operatorname{ent}_{3k}(\mathrm{id}_{1,\infty}) \leqslant \operatorname{ent}_k(\mathrm{id}_{1,p}) \operatorname{ent}_k(\mathrm{id}_{p,r}) \operatorname{ent}_k(\mathrm{id}_{r,\infty}). \quad \blacksquare$$

THEOREM 5. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric, normalized bases of E and F. Moreover, let* $\mathrm{id} \in L(E, F)$ *be the natural identity. Then we have*

$$\frac{1}{2e} \max_{l=k,\ldots,n} \frac{\lambda_F(l)}{\lambda_E(l)} \leqslant \operatorname{ent}_k(\mathrm{id}) \leqslant c \log^2(n/k + 1) \max_{l=k,\ldots,n} \frac{\lambda_F(l)}{\lambda_E(l)}$$
$$\text{for} \quad k = 1, \ldots, n, \tag{1}$$

and

$$\frac{1}{e} 2^{-k/n} \frac{\lambda_F(n)}{\lambda_E(n)} \leqslant \operatorname{ent}_k(\mathrm{id}) \leqslant c 2^{-k/n} \frac{\lambda_F(n)}{\lambda_E(n)} \qquad \text{for} \quad k \geqslant n, \tag{2}$$

where c denotes an absolute constant.

The following is a generalization of a lemma used by the author and N. Tomczak-Jaegerman.

LEMMA 6. *Let $\{e_i\}_{i=1}^n$ be a symmetric, normalized basis of E and $b_i \geqslant 0$, $i = 1,\dots, n$. Then we have*

$$c(\log(n/k+1))^{-1} \sum_{i=k+1}^{n} b_i \leqslant \left\| \sum_{i=k+1}^{n} b_i \lambda_E(i)^{-1} \sum_{j=1}^{i} e_j \right\| \quad \text{for} \quad k = 1,\dots, n,$$

where c is some absolute constant.

Proof. Without loss of generality we may assume that $k = 2^r$, $n = 2^s$, $r, s \in \mathbb{N}$.

Then

$$\left\| \sum_{i=k+1}^{n} b_i \lambda_E(i)^{-1} \sum_{j=1}^{i} e_j \right\| \geqslant \left\| \sum_{i=2^l+1}^{2^{l+1}} b_i \lambda_E(i)^{-1} \sum_{j=1}^{i} e_j \right\|$$

for all l with $r \leqslant l < s$. Thus we get

$$\left\| \sum_{i=k+1}^{n} b_i \lambda_E(i)^{-1} \sum_{j=1}^{i} e_j \right\| \geqslant \left\| \sum_{i=2^l+1}^{2^{l+1}} b_i \lambda_E(2^{l+1})^{-1} \sum_{j=1}^{2^l+1} e_j \right\| \geqslant \frac{1}{2} \sum_{i=2^l+1}^{2^{l+1}} b_i$$

for all l with $r \leqslant l < s$. Since there are only $s - r = \log n - \log k = \log n/k$ numbers l the estimate follows. ∎

Proof of Theorem 5. (2) is nothing but Lemma 4. The left-hand inequality of (1) follows from Lemma 4 and a factorization. Consider the injection $j_l \in L(E_l, E)$ and the projection $p_l \in L(F, F_l)$, where E_l and F_l denote the span of the first l unit vectors. Thus $p_l \operatorname{id} j_l$ gives the identity id_l from E_l onto F_l. Therefore

$$\operatorname{ent}_k(\operatorname{id}_l) = \operatorname{ent}_k(p_l \operatorname{id} j_l) \leqslant \|p_l\| \operatorname{ent}_k(\operatorname{id}) \|j_l\| = \operatorname{ent}_k(\operatorname{id}).$$

Thus, by Lemma 4,

$$\frac{1}{2e} \max_{l=k,\dots,n} \frac{\lambda_F(l)}{\lambda_E(l)} \leqslant e^{-1} \max_{l=1,\dots,n} 2^{-k/l} \frac{\lambda_F(l)}{\lambda_E(l)} \leqslant \operatorname{ent}_k(\operatorname{id}).$$

Now we prove the right-hand inequality. There are $\binom{n}{k}$ different subspaces of E that are spanned by exactly k unit vectors. For every subspace $[e_{i_1},\dots, e_{i_k}]$

there is a projection $p_{i_1,\ldots,i_k}$ with $p_{i_1,\ldots,i_k}(e_i)=e_i$ if $i\in\{i_1,\ldots,i_k\}$, the other unit vectors are mapped onto 0. Projection $q_{i_1,\ldots,i_k}$ is defined in the same way for $[f_{i_1},\ldots,f_{i_k}]$.

According to Lemma 4 we find 2^k vectors $x_1^{(i_1,\ldots,i_k)},\ldots,x_{2^k}^{(i_1,\ldots,i_k)}$ such that

$$\begin{aligned}\operatorname{id} p_{i_1,\ldots,i_k}(B_E) &= q_{i_1,\ldots,i_k}\operatorname{id} p_{i_1,\ldots,i_k}(B_E)\\ &\subseteq \bigcup_{i=1}^{2^k}\left\{x_i^{(i_1,\ldots,i_k)}+c\,\frac{\lambda_F(k)}{\lambda_E(k)}\,q_{i_1,\ldots,i_k}(B_F)\right\},\end{aligned}$$

where c is an absolute constant. Thus

$$\operatorname{id} p_{i_1,\ldots,i_k}(B_E)\subseteq \bigcup_{i=1}^{2^k}\left\{x_i^{(i_1,\ldots,i_k)}+c\,\frac{\lambda_F(k)}{\lambda_E(k)}\,B_F\right\}. \tag{3}$$

Now we claim that

$$\operatorname{id}(B_E)\subseteq \bigcup_{\{i_1,\ldots,i_k\}}\ \bigcup_{i=1}^{2^k}\left\{x_i^{(i_1,\ldots,i_k)}+\left(2c\log(n/k+1)\max_{k\leqslant l\leqslant n}\frac{\lambda_F(l)}{\lambda_E(l)}\right)B_F\right\}. \tag{4}$$

By this we proved that

$$\varepsilon_{2^k\binom{n}{k}}(\operatorname{id})\leqslant 2c\log(n/k+1)\max_{l=k,\ldots,n}\frac{\lambda_F(l)}{\lambda_E(l)},$$

and it is left to pass to dyadic entropy numbers. This is done as in the proof of Theorem 1 using also the triangle inequality. We prove now (4), suppose that $x\in B_E$ with $x=\sum_{i=1}^n a_ie_i$. Without loss of generality we may assume that $a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant 0$. With $b_i\geqslant 0$, $i=1,\ldots,n$, properly chosen we may write

$$\begin{aligned}x&=\sum_{i=1}^n b_i\lambda_E(i)^{-1}\sum_{j=1}^i e_j\\ &=\sum_{i=1}^k b_i\lambda_E(i)^{-1}\sum_{j=1}^i e_j+\sum_{i=k+1}^n b_i\lambda_E(i)^{-1}\sum_{j=1}^i e_j.\end{aligned}$$

Clearly, the first summand is contained in $p_{1,\ldots,k}(B_E)$, or to put it in other terms, according to (3) there is a vector $x_{i_0}^{1,\ldots,k}$ such that

$$\sum_{i=1}^k b_i\lambda_E(i)^{-1}\sum_{j=1}^i e_j\in\left\{x_{i_0}^{(1,\ldots,k)}+c\,\frac{\lambda_F(k)}{\lambda_E(k)}\,B_F\right\}.$$

The second summand satisfies

$$\sum_{i=k+1}^{n} b_i \lambda_E(i)^{-1} \sum_{j=1}^{i} e_j \in c \log(n/k+1) \max_{k \leqslant l \leqslant n} \frac{\lambda_F(l)}{\lambda_E(l)} B_F$$

because of Lemma 6. Thus (4) is obtained. ∎

THEOREM 7. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric, normalized bases of E and F. Suppose that $\lambda_F(k) = k^\alpha \lambda_E(k)$, $-1 \leqslant \alpha \leqslant 1$, and that $1/s > \max\{-\alpha, 0\}$ and $0 < t \leqslant \infty$. Then*

$$c^{-1} n^{\alpha+1/s} \leqslant L_{s,t}^e(\mathrm{id}_{E,F}) \leqslant c n^{\alpha+1/s},$$

where $c > 0$ is an absolute constant.

Proof. The left-hand inequality is easily obtained. According to (1) we have

$$\frac{1}{4e} n^\alpha = \frac{1}{4e} \frac{\lambda_F(n)}{\lambda_E(n)} \leqslant \mathrm{ent}_k(\mathrm{id}_{E,F}) \qquad \text{for} \quad k = [n/2], ..., n.$$

From this the left-hand inequality follows immediately. Now we prove the right-hand inequality. We consider two cases, first $\alpha \geqslant 0$. We get by Theorem 5 for some $c > 0$

$$c^{-1} L_{s,t}^e(\mathrm{id}_{E,F}) \leqslant c n^\alpha \left(\sum_{k=1}^{n} |k^{1/s-1/t} \log^2(n/k+1)|^t + \sum_{k=n+1}^{\infty} |k^{1/s-1/t} 2^{-k/n}|^t \right)^{1/t}.$$

By an elementary computation we get

$$L_{s,t}^e(\mathrm{id}_{E,F}) \leqslant c(s,t)\, n^{\alpha+1/s}.$$

In the other case, $\alpha < 0$, we get by Theorem 5

$$c^{-1} L_{s,t}^e(\mathrm{id}_{E,F}) \leqslant \left(\sum_{k=1}^{n} |k^{1/s-1/t+\alpha} \log^2(n/k+1)|^t + \sum_{k=n+1}^{\infty} |n^\alpha k^{1/s-1/t} \cdot 2^{-k/n}|^t \right)^{1/t}.$$

Again, by elementary computation the result follows. ∎

The following corollary is concerned with diagonal operators between spaces with symmetric bases. So, assume that $\{e_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ are symmetric bases of Banach spaces E and F. Then $D \in L(E, F)$ is called a diagonal operator if $D(e_i) = d_i f_i$, $i \in \mathbb{N}$, with $d_i \in \mathbb{R}$.

COROLLARY 8. *Let* $\{e_i\}_{i=1}^{\infty}$ *and* $\{f_i\}_{i=1}^{\infty}$ *be symmetric, normalized bases of* E *and* F. *Suppose that* $\lambda_F(k) = k^{\alpha}\lambda_E(k)$, $-1 \leqslant \alpha \leqslant 1$, $1/r > \max\{\alpha, 0\}$, $0 < t \leqslant \infty$, *and* $1/s = 1/r - \alpha$. *Then*

$$D \in \mathscr{L}_{s,t}^{(e)}(E, F) \qquad \textit{if and only if} \quad (d_i)_{i=1}^{\infty} \in l^{r,t}.$$

This corollary follows from Theorem 7 and the proof of Theorem 2 in [1]. We also use that $\lambda_E(n)\,\lambda_{E^*}(n) = n$.

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